On the rank of payoff matrices with long-term assets

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2011.84

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://centredeconomiesorbonne.univ-paris1.fr/bandeau-haut/documents-de-travail/
ISSN : 1955-611X

On the rank of payoff matrices with long-term assets.

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December 2011

Abstract

We consider a stochastic financial exchange economy with a finite date-event tree representing time and uncertainty and a nominal financial structure with possibly long-term assets. We exhibit a sufficient condition under which the payoff matrix and the full payoff matrix have the same rank. This generalizes previous results of Angeloni-Cornet and Magill-Quinzii involving only short-term assets. We then derive existence results with assumptions only based on the fundamentals of the economy.

Keywords: Incomplete Markets, financial equilibrium, multi-period model, long-term assets.

JEL codes: D5, D4, G1

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Contents

1 Introduction

In this paper, we consider a standard model of stochastic financial exchange economy with a finite date-event tree representing time and uncertainty as described in [4]. The financial structure is composed of a finite set of nominal assets with possibly long-term assets.

The financial structure is formally represented by the payoff matrix, which provides for each asset and each node, the return of this asset in this node. For a simpler presentation, we adopt the convention of Angeloni and Cornet [1] where an asset is issued at one date and never retraded after. We refer to this article to show how to transform a financial structure with possible retrading as considered in [4] into an equivalent financial structure without retrading.

When we consider the asset markets at different dates and nodes and the asset prices on these markets, we need to define a full payoff matrix to determine the budget constraints of an agent. The full payoff matrix is derived from the payoff matrix by adding on the column corresponding to an asset and on the row corresponding to the issuance node of this asset, the opposite of the price of the asset on the financial market.

Now, with the full payoff matrix, we can define arbitrage free prices and characterize them by the existence of positive state prices such that the asset prices are the present value of the payoffs.

In a two period economy, it is well known and actually quite obvious that the rank of the payoff and full payoff matrices are equal when the asset prices do not exhibit arbitrage. This means that the ranges of the possible wealth transfers with the payoff and the full payoff matrices have the same dimension. In particular, we can characterize a complete financial structure only by knowing the rank of the payoff matrix. Indeed, if the rank is equal to the number of states in the second period, whatever are the arbitrage free asset prices, the full payoff matrix is also of maximal rank and then all transfers compatible with the state prices are feasible through the financial structure.

With more than two periods, the above result is no more true as shown in [4, 1]. Below we provide a simple numerical example where the payoff matrix has a maximal rank and the full payoff matrix does not for some arbitrage free asset prices. So, for some asset prices, the financial structure is complete and for some others it is not. It is then no more possible to characterize a complete financial structure only with the payoff matrix.

Furthermore, the dependance of the rank of the full payoff matrix from arbitrage free asset prices may lead to the failure of the existence of a financial equilibrium. See an example in [4]. In a two period economy, this kind of phenomenon appears with real assets due to the drop of the rank on the payoff matrix but never with nominal assets.

Actually, in [4] and [1], it is shown that the ranks of the payoff and full payoff matrices are equal when the assets are all short term. A short term asset is characterized by the fact that it has non zero return only at the immediate successors of its issuance node. If an asset is not short term, it is called a long term asset.

Our main purpose is this paper is to tackle the question of the ranks of the return matrices with long term assets and to obtain existence result of financial equilibria under assumptions more tractable than those of [1].

After introducing notations and the model of a financial structure in Section 2, we provide a sufficient condition, Assumption R, in Section 3, to get the equality of the ranks of the payoff and full payoff matrices for non arbitrage asset prices. We show that Assumption **(R)** is satisfied if all assets are short term, if there is a unique issuance date, or if there is no overlap of the nodes with non zero returns for two different assets. More generally, Assumption **(R)** translates the fact that the assets issued at a given node are true financial innovations in the sense that the payoffs cannot be replicated by assets issued before.

In Section 4, we consider a stochastic financial exchange economy with possibly long term nominal assets and we provide several existence results when Assumption **(R)** is satisfied by the payoff matrix. These results are based on the existence result (Theorem 3.1) of [1] but Assumption **(R)** allows us to replace an abstract assumption in [1] by a more verifiable assumption on the return matrix. Furthermore, our result then holds true for any state prices.

In [1], it is mentioned that the abstract condition is satisfied with only short term assets. So our contribution could be seen as the extension to long term assets under Assumption R.

The differences between our different results come from the fact that we can consider more general portfolio sets translating restricted participations when the return matrix is of maximal rank whereas when we have redundant assets, we get only existence when the agents have a full access to the asset markets.

2 The T-period financial exchange economy

In this section, we present the model and the notations, which are borrowed from Angeloni-Cornet[1] and are essentially the same as those of Magill-Quinzii[4].

2.1 Time and uncertainty

We[1] consider a multi-period exchange economy with $(T+1)$ dates, $t \in \mathcal{T} := \{0, ..., T\}$, and a finite set of agents $\mathcal{I}$. The uncertainty is described by a date-event tree $\mathbb{D}$ of length $T+1$. The set $\mathbb{D}_t$ is the set of nodes (also called date-

[1]We use the following notations. A $(\mathbb{D} \times \mathcal{J})$-matrix A is an element of $\mathbb{R}^{\mathbb{D}\times\mathcal{J}}$, with entries $(a^j_\xi)_{(\xi\in\mathbb{D},j\in\mathcal{J})}$; we denote by $A_\xi \in \mathbb{R}^{\mathcal{J}}$ the ξ-th row of A and by $A^j \in \mathbb{R}^{\mathbb{D}}$ the j-th column of A. We recall that the transpose of A is the unique $(\mathcal{J} \times \mathbb{D})$-matrix ${}^t\mathrm{A}$ satisfying $(Ax) \bullet_{\mathbb{D}} y = x \bullet_{\mathcal{J}} ({}^t\mathrm{Ay})$ for every $x \in \mathbb{R}^{\mathcal{J}}$, $y \in \mathbb{R}^{\mathbb{D}}$, where $\bullet_{\mathbb{D}}$ [resp. $\bullet_{\mathcal{J}}$] denotes the usual inner product in $\mathbb{R}^{\mathbb{D}}$ [resp. $\mathbb{R}^{\mathcal{J}}$]. We denote by rank$A$ the rank of the matrix A and by Vect (A) the range of A, that is the linear sub-space spanned by the column vectors of A. For every subset $\tilde{\mathbb{D}} \subset \mathbb{D}$ and $\tilde{\mathcal{J}} \subset \mathcal{J}$, the matrix $A^{\tilde{\mathcal{J}}}_{\tilde{\mathbb{D}}}$ is the $(\tilde{\mathbb{D}}\times\tilde{\mathcal{J}})$-sub-matrix of A with entries a^j_ξ for every $(\xi, j) \in (\tilde{\mathbb{D}}\times\tilde{\mathcal{J}})$. Let x, y be in $\mathbb{R}^n$; $x \geq y$ (resp. $x \gg y$) means $x_h \geq y_h$ (resp. $x_h > y_h$) for every $h = 1, \ldots, n$ and we let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$, $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x \gg 0\}$. We also use the notation $x > y$ if $x \geq y$ and $x \neq y$. The Euclidean norm in the Euclidean different spaces is denoted $\|.\|$ and the closed ball centered at x and of radius $r > 0$ is denoted $\bar{B}(x, r) := \{y \in \mathbb{R}^n \mid \|y - x\| \leq r\}$.

events) that could occur at date t and the family $(\mathbb{D}_t)_{t\in\mathcal{T}}$ defines a partition of the set $\mathbb{D}$; for each $\xi \in \mathbb{D}$, we denote by $t(\xi)$ the unique date $t \in \mathcal{T}$ such that $\xi \in \mathbb{D}_t$.

At date $t = 0$, there is a unique node ξ_0, that is $\mathbb{D}_0 = \{\xi_0\}$. As $\mathbb{D}$ is a tree, each node ξ in $\mathbb{D}\setminus\{\xi_0\}$ has a unique immediate predecessor denoted $pr(\xi)$ or ξ^-. The mapping pr maps $\mathbb{D}_t$ to $\mathbb{D}_{t-1}$. Each node $\xi \in \mathbb{D}\setminus\mathbb{D}_T$ has a set of immediate successors defined by $\xi^+ = \left\{\bar{\xi} \in \mathbb{D} : \xi = \bar{\xi}^-\right\}$.

For $\tau \in \mathcal{T} \setminus \{0\}$ and $\xi \in \mathbb{D} \setminus \cup_{t=0}^{\tau-1}\mathbb{D}_t$, we define $pr^\tau(\xi)$ by the recursive formula: $pr^\tau(\xi) = pr\left(pr^{\tau-1}(\xi)\right)$. We then define the set of successors and the set of predecessors of ξ as follows:

$$\mathbb{D}^+(\xi) = \{\xi' \in \mathbb{D} : \exists\tau \in \mathcal{T} \setminus \{0\} \mid \xi = pr^\tau(\xi')\}$$
$$\mathbb{D}^-(\xi) = \{\xi' \in \mathbb{D} : \exists\tau \in \mathcal{T} \setminus \{0\} \mid \xi' = pr^\tau(\xi)\}$$

If $\xi' \in \mathbb{D}^+(\xi)$ [resp. $\xi' \in \mathbb{D}^+(\xi) \cup \{\xi\}$], we shall use the notation $\xi' > \xi$ [resp. $\xi' \geq \xi$]. Note that $\xi' \in \mathbb{D}^+(\xi)$ if and only if $\xi \in \mathbb{D}^-(\xi')$ and similarly $\xi' \in \xi^+$ if and only if $\xi = (\xi')^-$.

2.2 The financial structure

The financial structure is constituted by a finite set of assets denoted $\mathcal{J} = \{1, \ldots, J\}$. An asset $j \in \mathcal{J}$ is a contract issued at a given and unique node in $\mathbb{D}$ denoted $\xi(j)$, called issuance node of j. Each asset is bought or sold only at its issuance node $\xi(h)$ and yields payoffs only at the successor nodes ξ' of $\mathbb{D}^+(\xi(j))$. To simplify the notation, we consider the payoff of asset j at every node $\xi \in \mathbb{D}$ and we assume that it is zero if ξ is not a successor of the issuance node $\xi(j)$. The payoff may depend upon the spot price vector $p \in \mathbb{R}^{\mathbb{L}}$ and is denoted by $V_\xi^j(p)$. Formally, we assume that $V_\xi^j(p) = 0$ if $\xi \notin \mathbb{D}^+(\xi(j))$.

For each consumer, $z_i = (z_i^j)_{j\in\mathcal{J}} \in \mathbb{R}^{\mathcal{J}}$ is called the portfolio of agent i. If $z_i^j > 0$ [resp. $z_i^j < 0$], then $|z_i^j|$ is the quantity of asset j bought [resp. sold] by agent i at the issuance node $\xi(j)$.

We assume that each consumer i is endowed with a portfolio set $Z_i \subset \mathbb{R}^{\mathcal{J}}$, which represents the set of admissible portfolios for agent i. For a discussion on this concept we refer to Angeloni-Cornet [1], Aouani-Cornet [2] and the references therein.

To summarize a financial structure $\mathcal{F} = \left(\mathcal{J}, (Z_i)_{i\in\mathcal{I}}, (\xi(j))_{j\in\mathcal{J}}, V\right)$ consists of

- a set of assets $\mathcal{J}$,
- a collection of portfolio sets $(Z_i \subset \mathbb{R}^{\mathcal{J}})_{i\in\mathcal{I}}$,
- a node of issuance $\xi(j)$ for each asset $j \in \mathcal{J}$,
- a payoff mapping $V : \mathbb{R}^{\mathbb{L}} \to \mathbb{R}^{\mathbb{D}\times\mathcal{J}}$ which associates to every spot price $p \in \mathbb{R}^L$ the $(\mathbb{D} \times \mathcal{J})$-payoff matrix $V(p) = \left(V_\xi^j(p)\right)_{\xi\in\mathbb{D}, j\in\mathcal{J}}$ and satisfies the condition $V_\xi^j(p) = 0$ si $\xi \notin \mathbb{D}^+(\xi(j))$.

The price of asset j is denoted by q_j; it is paid at its issuance node $\xi(j)$. We let $q = (q_j)_{j\in\mathcal{J}} \in \mathbb{R}^{\mathcal{J}}$ be the asset price vector.

The full payoff matrix $W(p,q)$ is the $(\mathbb{D}\times\mathcal{J})$-matrix with the following entries:

$$W_\xi^j(p,q) := V_\xi^j(p) - \delta_{\xi,\xi(j)} q_j,$$

where $\delta_{\xi,\xi'} = 1$ if $\xi = \xi'$ and $\delta_{\xi,\xi'} = 0$ otherwise.

So, given the prices (p,q), the full flow of returns for a given portfolio $z \in \mathbb{R}^{\mathcal{J}}$ is $W(p,q)\,z$ and the full return at node ξ is

$$\begin{aligned}[W(p,q)\,z](\xi) &:= W_\xi(p,q) \bullet_{\mathcal{J}} z = \sum_{j\in\mathcal{J}} V_\xi^j(p)\,z_j - \sum_{j\in\mathcal{J}} \delta_{\xi,\xi(j)} q_j z_j \\ &= \sum_{\{j\in\mathcal{J}\ \mid\ \xi(j)<\xi\}} V_\xi^j(p)\,z_j - \sum_{\{j\in\mathcal{J}\ \mid\ \xi(j)=\xi\}} q_j z_j,\end{aligned}$$

We now recall that for a given spot price p, the asset price q is an arbitrage free price if it does not exist a portfolio $z \in \mathbb{R}^{\mathcal{J}}$ such that $W(p,q)z > 0$. q is an arbitrage free price if and only if it exists a so-called state price vector $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$ such that ${}^tW(p,q)\lambda = 0$ (see, e.g. Magill-Quinzii [4]). Taken into account the particular structure of the matrix $W(p,q)$, this is equivalent to

$$\forall j \in \mathcal{J}, \lambda_{\xi(j)} q_j = \sum_{\xi\in\mathbb{D}^+(\xi(j))} \lambda_\xi V_\xi^j(p).$$

3 On the ranks of the return matrices

In a two period model, the rank of $V(p)$ and of $W(p,q)$ are equals when the price q is a arbitrage free price. Indeed, the matrix $W(p,q)$ is simply built from $V(p)$ by replacing the first row by the transpose of $-q$. Then, since the no-arbitrage condition implies that q is a positive linear combination of the rows of $V(p)$, we easily conclude.

As already noticed in Magill and Quinzii [4] and in Angeloni and Cornet [1], this result is no more true if the number of dates is strictly larger than 2. Let us give an example with a maximal rank return matrix. Let a financial structure with three dates, each non-terminal node has two immediate successors so that $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}\}$. At each non-terminal node, two assets are issued, hence $J = 6$. The return matrix V is constant and equal to

$$\mathbf{V} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

One remarks that the rank of the matrix V is 6. We now consider the asset price $q = (7,7,2,1,1,1)$. q is a arbitrage free price since ${}^tW(q)\,\lambda = 0$ with $\lambda = (1,1,1,1,1,1,1) \in \mathbb{R}^7_{++}$. Hence the full-return matrix is

$$\mathbf{W}(\mathbf{q}) = \begin{bmatrix} -7 & -7 & 0 & 0 & 0 & 0 \\ 1 & 2 & -2 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The rank of $W(q)$ is 5 since the dimension of the kernel of $W(q)$ is 1. So, even if the rank of V is maximal, then for the arbitrage free price q, the financial market is incomplete in the sense that the rank of the full return matrix is strictly lower than $\#\mathbb{D}-1$. Then the rank of V is no more sufficient to determine if the market is complete or not and the completeness may depend on the asset price which is endogenously chosen by the market.

Our next example exhibits a converse paradox. Indeed the return matrix V is not of maximal rank but the market is complete for a well chosen arbitrage free price q. The date-event tree is the same as in the previous example and the number of assets as well as the dates of issuance are also identical. But now the return matrix is

$$\mathbf{V} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

One easily check that the rank of V is 5 strictly smaller than the maximal rank which is 6. Let $q = \left(\frac{5}{2}, \frac{5}{2}, 1, 2, 2, 1\right)$. The price q is arbitrage free since ${}^tW\,(q)\,\lambda = 0$ with $\lambda = (1, 1/2, 1/2, 1/2, 1, 1, 1/2)$. The full return matrix is then:

$$\mathbf{W}(\mathbf{q}) = \begin{bmatrix} -\frac{5}{2} & -\frac{5}{2} & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -2 & 0 & 0 \\ 1 & 1 & 0 & 0 & -2 & -1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

But the rank of $W(q)$ is 6, that is the maximal rank and then the market is complete.

The discrepancy of the ranks between the return matrix and the full return matrix is not only a problem to determine what is a complete market for a multi-period economy but, as shown in an example of Magill-Quinzii [4], it also leads to the non-existence of equilibrium even with a nominal asset structure where the return matrix V does not depend on the spot price p.

In Magill-Quinzii [4] and in Angeloni-Cornet [1], it is shown that the discrepancy of the ranks cannot happen when there are only short-lived assets, that is the payoff of an asset j issued at node ξ is zero except at the immediate successor of ξ.

3.1 Equality between ranks of payoff matrices

The main purpose of this section is to provide a sufficient condition compatible with long term assets under which the rank of the return matrix is equal to the rank of the full return matrix for every arbitrage free asset price.

We first introduce some additional notations. For all $\xi \in \mathbb{D} \setminus \mathbb{D}_T$, $\mathcal{J}(\xi)$ is the set of assets issued at the node ξ, that is $\mathcal{J}(\xi) = \{j \in \mathcal{J} \mid \xi(j) = \xi\}$ and $\mathcal{J}(\mathbb{D}^-(\xi))$ is the set of assets issued at a predecessor of ξ, that is $\mathcal{J}(\mathbb{D}^-(\xi)) = \{j \in \mathcal{J} \mid \xi(j) < \xi\}$. For all $t \in \{0, \ldots, T-1\}$, we denote by $\mathcal{J}_t$ the set of assets issued at date t, that is, $\mathcal{J}_t = \{j \in \mathcal{J} \mid \xi(j) \in \mathbb{D}_t\}$.

Let $(\tau_1, \ldots, \tau_k)$ such that $0 \leq \tau_1 < \tau_2 < \ldots < \tau_k \leq T-1$ be the dates at which there is at least the issuance of one asset, that is $\mathcal{J}_{\tau_\kappa} \neq \emptyset$. For $\kappa = 1, \ldots, k$, let $\mathbb{D}^e_{\tau_\kappa}$ be the set of nodes at date τ_κ at which there is the issuance of at least one asset. $\mathbb{D}^e = \cup_{\kappa=1}^k \mathbb{D}^e_{\tau_\kappa}$ is the set of nodes at which there is the issuance of at least one asset. We remark that

$$\bigcup_{\tau \in \{0,\ldots,T-1\}} \mathcal{J}_\tau = \bigcup_{\kappa \in \{1,\ldots,k\}} \mathcal{J}_{\tau_\kappa} = \mathcal{J}, \quad J = \sum_{\kappa \in \{1,\ldots,k\}} \#\mathcal{J}_{\tau_\kappa}$$

and for all $\tau \in \{\tau_1, \ldots, \tau_k\}$, $\bigcup_{\xi \in \mathbb{D}_\tau} \mathcal{J}(\xi) = \mathcal{J}_\tau$.

In the remainder of this section, we consider a fixed spot price p so that we remove it as an argument of V for the sake of simpler notations.

We now state our assumption on the payoff matrix and its consequence on the rank of the matrix V and $W(q)$.

Assumption (R). $\forall \kappa \in \{2, \ldots, k\}$, $\forall \xi \in \mathbb{D}^e_{\tau_\kappa}$,

$$\mathrm{Vect}\left(V_{\mathbb{D}^+(\xi)}^{\mathcal{J}(\mathbb{D}^-(\xi))}\right) \bigcap \mathrm{Vect}\left(V_{\mathbb{D}^+(\xi)}^{\mathcal{J}(\xi)}\right) = \{0\}.$$

This assumption means that the returns of the assets issued at a node ξ are not redundant with the returns of the assets issued at a predecessor node of ξ. So, the issuance of additional assets at ξ are a true financial innovation since the payoffs in the successors of ξ cannot be replicated by the payoffs of a portfolio built with the assets issued before ξ.

In the following lemma, we show that if Assumption **(R)** holds true for the financial structure $\mathcal{F}$, it is also true for any financial substructure $\mathcal{F}'$ of $\mathcal{F}$ obtained by considering only a subset $\mathcal{J}'$ of the set of asset $\mathcal{J}$.

Lemma 1. *Let*

$$\mathcal{F} = \left(\mathcal{J}, (Z_i)_{i \in \mathcal{I}} \left(\xi(j)\right)_{j \in \mathcal{J}}, V\right) \quad \mathcal{F}' = \left(\mathcal{J}', (Z'_i)_{i \in \mathcal{I}} \left(\xi(j)\right)_{j \in \mathcal{J}'}, V'\right)$$

*two financial structures such as $\mathcal{J}' \subset \mathcal{J}$. If Assumption **R** holds true for the structure $\mathcal{F}$ then it holds also true for the structure $\mathcal{F}'$.*

Proof. The proof of Lemma 1 is given in Appendix.

Remark 1. *The converse of Lemma 1 is not true. Let us consider an economy with three periods such as:* $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}\}$. *There are three assets issued at nodes* 0, 1 *and* 2. *The return matrix is:*

$$V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

One remark that Assumption **(R)** *is not satisfied for* V *whereas it holds true for the reduced financial structure where we keep only the two first assets and for which the return matrix is*

$$V = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The next proposition provides some sufficient conditions under which Assumption **(R)** holds true.

Proposition 1. *The return matrix* V *satisfies Assumption* **(R)** *if one of the following condition is satisfied:*

(i) For all $j \in \mathcal{J}$, *asset* j *is a short term asset in the sense that* $V^j_{\xi'} = 0$ *if* $\xi' \notin \xi^+$.

(ii) All assets are issued at the same date τ_1.

(iii) For all $\xi \in \mathbb{D}^e$, $\mathbb{D}^+(\xi) \cap \mathbb{D}^e = \emptyset$, *which means that if an asset is issued at node* ξ, *then no assets is issued at a successor of* ξ.

(iv) For all $(\xi, \xi') \in (\mathbb{D}^e)^2$, *if* $\xi < \xi'$, *then* $V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi')} = 0$, *which means that if an asset* j *is issued at node* ξ *and another one at a successor* ξ', *then the return of* j *at the successors of* ξ' *are equal to* 0.

The proof of this proposition is left to the reader. It is a consequence of the fact that either $\mathcal{J}(\mathbb{D}^-(\xi))$ is an empty set or $\mathrm{Vect}(V^{\mathcal{J}(\mathbb{D}^-(\xi))}_{\mathbb{D}^+(\xi)}) = \{0\}$.

Now, we state the main result of this section:

Proposition 2. *If the return matrix* V *satisfies Assumption* **(R)**, *then for all arbitrage free price* q, $\mathrm{rank}V = \mathrm{rank}W(q)$.

Condition (i) of Proposition 1 shows that Proposition 2 is a generalization of Proposition 5.2. b) and c) in Angeloni-Cornet [1] and of Magill-Quinzii [4] where only short-term assets are considered.

Remark 2. *For the following financial structure, Assumption* **(R)** *does not hold true and yet, for any (arbitrage free or not) price of assets q,* $\mathrm{rank}V = \mathrm{rank}W(q)$. *So Assumption* **(R)** *is sufficient but not necessary. Let us consider the date-event tree $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}\}$. Three assets are issued, two at ξ_0 and one at ξ_1. For all non-arbitrage price $q = (q_1, q_2, q_3)$, the return matrix and the full return matrix are:*

$$\mathbf{V} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad \mathbf{W}(\mathbf{q}) = \begin{bmatrix} -q_1 & -q_2 & 0 \\ 1 & 0 & -q_3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

One easily checks $\mathrm{rank}V = \mathrm{rank}W(q) = 3$ *whatever is the asset price q.*

Remark 3. *In Magill and Quinzii [4], it is assumed that a long-term asset is re-traded at each nodes after its issuance node. In Angeloni and Cornet [1], it is shown that a financial structure with re-trading is equivalent to a financial without re-trading by considering that a re-trade is equivalent to the issuance of a new asset.*

We remark that if the financial structure has long-term assets with re-trading, then Assumption **(R)** *is not satisfied by the equivalent financial structure without re-trading. Let us give an example. Let us consider the date-event tree $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}\}$. Two assets are issued at ξ_0 with dividend processes*

$$V^1 = (0, (0,0), (1,0,1,0)) \qquad V^2 = (0, (0,0), (0,1,0,1))$$

If these two assets are re-traded at each non-terminal node successor of ξ_0, for all arbitrage free price $q = (q_1(\xi_0), q_2(\xi_0), q_1(\xi_1), q_2(\xi_1), q_1(\xi_2), q_2(\xi_2))$, the full payoff matrix is:

$$\mathbf{W_{MQ}}(\mathbf{q}) = \begin{bmatrix} -q_1(\xi_0) & -q_2(\xi_0) & 0 & 0 & 0 & 0 \\ q_1(\xi_1) & q_2(\xi_1) & -q_1(\xi_1) & -q_2(\xi_1) & 0 & 0 \\ q_1(\xi_2) & q_2(\xi_2) & 0 & 0 & -q_1(\xi_2) & -q_2(\xi_2) \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

But if, following the methodology of Angeloni-Cornet [1], we consider an equivalent financial structure with 6 *assets without retrading, we obtain the following full payoff matrix with $\tilde{q} = (\tilde{q}_1, \tilde{q}_2, \tilde{q}_{11}, \tilde{q}_{12}, \tilde{q}_{21}, \tilde{q}_{22})$,*

$$\mathbf{W_{AC}}(\tilde{\mathbf{q}}) = \begin{bmatrix} -\tilde{q}_1 & -\tilde{q}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\tilde{q}_{11} & -\tilde{q}_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\tilde{q}_{21} & -\tilde{q}_{22} \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We remark that the two financial structures are equivalent when $q = \tilde{q}$ since, by performing elementary operations on the columns of $W_{AC}(q)$, we obtain $W_{MQ}(q)$. Assumption **(R)** *is not satisfied because the returns on assets issued at nodes ξ_1 and ξ_2 are redundant with the return of assets issued to node ξ_0. As already remarked in Magill-Quinzii [4], the rank of the full payoff matrix $W_{MQ}(q)$ depends on the asset price vector q.*

Proof of Proposition 2

For all $\xi \in \mathbb{D}^e$, we denote by $n(\xi)$ the number of assets issued at this node and by $\mathrm{rk}(\xi)$ the rank of $V_{\mathbb{D}}^{\mathcal{J}(\xi)}$. We also simplify the notation by defining $V^{\mathcal{J}(\xi)} := V_{\mathbb{D}}^{\mathcal{J}(\xi)}$ and $W^{\mathcal{J}(\xi)}(q) := W_{\mathbb{D}}^{\mathcal{J}(\xi)}(q)$. We assume without any loss of generality that the columns of V are ranked in such a way that the $\mathrm{rk}(\xi)$ first columns of $V^{\mathcal{J}(\xi)}$ are linearly independent.

Step 1: For all $\xi \in \mathbb{D}^e$, $\mathrm{rank} W^{\mathcal{J}(\xi)}(q) = \mathrm{rk}\,(\xi)$.

If $\mathrm{rk}(\xi) = n(\xi)$, $\mathrm{rank}W^{\mathcal{J}(\xi)}(q) = n(\xi)$. Indeed, since $\mathrm{rk}(\xi) = n(\xi)$, there exists a regular $n(\xi)$ square sub-matrix of $V^{\mathcal{J}(\xi)}$. Since $W^{\mathcal{J}(\xi)}(q)$ is obtained from $V^{\mathcal{J}(\xi)}$ by replacing a zero row by the row of asset prices issued at ξ, the regular $n(\xi)$ square sub-matrix is also a sub-matrix of $W^{\mathcal{J}(\xi)}(q)$, hence the rank of $W^{\mathcal{J}(\xi)}(q)$ is higher or equal to $n(\xi)$. But, since $n(\xi)$ is the number of columns of $W^{\mathcal{J}(\xi)}(q)$, then its rank is lower or equal to $n(\xi)$ so that we obtain the desired result[2].

If $\mathrm{rk}(\xi) < n(\xi)$, let us consider $\lambda = (\lambda_\xi)_{\xi \in \mathbb{D}} \in \mathbb{R}^{\mathbb{D}}_{++}$ such that ${}^tW\,(q)\,\lambda = 0$. Such λ exists since q is a arbitrage free price.

Let $\{j_1, \ldots, j_{\mathrm{rk}(\xi)}, j_{\mathrm{rk}(\xi)+1}, \ldots, j_{n(\xi)}\}$ be the assets issued at date ξ with $(V^{j_\ell})_{\ell=1}^{\mathrm{rk}(\xi)}$ linearly independent. By the same argument as above, $(W^{j_\ell}(q))_{\ell=1}^{\mathrm{rk}(\xi)}$ are also linearly independent. Hence the rank of $W^{\mathcal{J}(\xi)}(q)$ is larger or equal to $\mathrm{rk}(\xi)$. Let us now prove that the rank of $W^{\mathcal{J}(\xi)}(q)$ is not strictly larger than $\mathrm{rk}(\xi)$. It suffices to prove that for all $\nu = \mathrm{rk}(\xi) + 1, \ldots, n(\xi)$, $W^{j_\nu}(q) \in \mathrm{Vect}((W^{j_\ell}(q))_{\ell=1}^{\mathrm{rk}(\xi)})$.

Since the rank of $V^{\mathcal{J}(\xi)}$ is $\mathrm{rk}(\xi)$, V^{j_ν} is a linear combination of $(V^{j_\ell})_{\ell=1}^{\mathrm{rk}(\xi)}$, hence there exists $(\alpha_\ell)_{\ell=1}^{\mathrm{rk}(\xi)}$ such that $\sum_{\ell=1}^{\mathrm{rk}(\xi)} \alpha_\ell V^{j_\ell} = V^{j_\nu}$. Since ${}^tW\,(q)\,\lambda = 0$, $\lambda_\xi q_{j_\nu} = \sum_{\xi' > \xi} \lambda_{\xi'} V_{\xi'}^{j_\nu}$. Hence $\lambda_\xi q_{j_\nu}$ is equal to

$$\begin{aligned} \textstyle\sum_{\xi'>\xi} \left[\lambda_{\xi'} \sum_{\ell=1}^{\mathrm{rk}(\xi)} \alpha_\ell V_{\xi'}^{j_\ell}\right] &= \textstyle\sum_{\ell=1}^{\mathrm{rk}(\xi)} \left[\alpha_\ell \sum_{\xi'>\xi} \lambda_{\xi'} V_{\xi'}^{j_\ell}\right] \\ &= \textstyle\sum_{\ell=1}^{\mathrm{rk}(\xi)} \left[\alpha_\ell \lambda_\xi q_{j_\ell}\right] = \lambda_\xi \sum_{\ell=1}^{\mathrm{rk}(\xi)} \alpha_\ell q_{j_\ell} \end{aligned}$$

[2] Note that we do not use the fact that the asset price is an arbitrage free price in this part of the proof.

Hence $q_{j_\nu} = \sum_{\ell=1}^{\mathrm{rk}(\xi)} \alpha_\ell q_{j_\ell}$, which together with $\sum_{\ell=1}^{\mathrm{rk}(\xi)} \alpha_\ell V^{j_\ell} = V^{j_\nu}$ imply that $\sum_{\ell=1}^{\mathrm{rk}(\xi)} \alpha_\ell W^{j_\ell}(q) = W^{j_\nu}(q)$. So $W^{j_\nu}(q) \in \mathrm{Vect}((W^{j_\ell}(q))_{\ell=1}^{\mathrm{rk}(\xi)})$.

For $\kappa = 1, \ldots, k$, we let $\mathrm{rk}_\kappa = \sum_{\xi \in \mathbb{D}^e_{\tau_\kappa}} \mathrm{rk}\,(\xi)$.

Step 2: $\forall \kappa \in \{1, \ldots, k\}\,, \mathrm{rank} V^{\mathcal{J}_{\tau_\kappa}} = \sum_{\xi \in \mathbb{D}^e_{\tau_\kappa}} \mathrm{rk}(\xi) = \mathrm{rk}_\kappa$ and $\mathrm{rank} W^{\mathcal{J}_{\tau_\kappa}}(q) = \sum_{\xi \in \mathbb{D}^e_{\tau_\kappa}} \mathrm{rank} W^{\mathcal{J}(\xi)}(q) = \mathrm{rk}_\kappa$

If $\#\mathbb{D}^e_{\tau_\kappa} = 1$, this coincides with what is proved in Step 1. If $\#\mathbb{D}^e_{\tau_\kappa} > 1$, let $\xi \in \mathbb{D}^e_{\tau_\kappa}$. Then

$$\left[\sum_{\{\xi' \in \mathbb{D}^e_{\tau_\kappa} \setminus \{\xi\}\}} \mathrm{Vect}\left(V^{\mathcal{J}(\xi')}\right)\right] \bigcap \mathrm{Vect}\left(V^{\mathcal{J}(\xi)}\right) = \{0\}$$

Indeed, the return of the asset $j \in \mathcal{J}(\xi)$ can be non zero only on the subtree $\mathbb{D}^+(\xi)$, whereas for the asset $j \in \mathcal{J}(\xi')$ for $\xi' \in \mathbb{D}^e_{\tau_\kappa} \setminus \{\xi\}$, the returns on the subtree $\mathbb{D}^+(\xi)$ are identically equal to 0. This implies that the subspaces $\left(\mathrm{Vect}\left(V^{\mathcal{J}(\xi')}\right)\right)_{\xi' \in \mathbb{D}^e_{\tau_\kappa}}$ are in direct sum so, using Step 1, we get the following formula for the dimensions:

$$\dim \mathrm{Vect}\left(V^{\mathcal{J}_{\tau_\kappa}}\right) = \sum_{\xi \in \mathbb{D}^e_{\tau_\kappa}} \dim \mathrm{Vect}\left(V^{\mathcal{J}(\xi)}\right) = \sum_{\xi \in \mathbb{D}^e_{\tau_\kappa}} \mathrm{rk}\,(\xi) = \mathrm{rk}_\kappa$$

For the matrix $W(q)$, the proof is the same as above if we remark that the full return of an asset $j \in \mathcal{J}(\xi)$ can be non zero only on the subtree $\xi \cup \mathbb{D}^+(\xi)$. Hence if ξ and ξ' are two different issuance nodes in $\mathbb{D}^e_{\tau_\kappa}$, there is no node ξ'' such that the coordinates of a column vectors of the matrix $W^{\mathcal{J}(\xi)}(q)$ and of a column vector of the matrix $W^{\mathcal{J}(\xi')}(q)$ are both non zero. Hence we get the following formula from which the result is a direct consequence of Step 1:

$$\left[\sum_{\{\xi' \in \mathbb{D}^e_{\tau_\kappa} \setminus \{\xi\}\}} \mathrm{Vect}\left(W^{\mathcal{J}(\xi')}(q)\right)\right] \bigcap \mathrm{Vect}\left(W^{\mathcal{J}(\xi)}(q)\right) = \{0\}$$

Step 3. $\mathrm{rank} V = \sum_{\kappa=1}^k \mathrm{rank} V^{\mathcal{J}_{\tau_\kappa}} = \sum_{\kappa=1}^k \mathrm{rk}_\kappa$ and $\mathrm{rank} V = \mathrm{rank} W\,(q) = \sum_{\kappa=1}^k \mathrm{rank} W^{\mathcal{J}_{\tau_\kappa}}(q)$.

We first remark that $\mathrm{Vect}\,(V) = +_{\kappa=1}^k \mathrm{Vect}\left(V^{\mathcal{J}_{\tau_\kappa}}\right)$ which implies using Step 2 that $\mathrm{rank} V \leq \sum_{\kappa \in \{1,\ldots,k\}} \mathrm{rank} V^{\mathcal{J}_{\tau_\kappa}} = \sum_{\kappa=1}^k \mathrm{rk}_\kappa$.

We remark that if $k = 1$, then the result is obvious. If $k > 1$, we will prove that the rank of V is equal to $\sum_{\kappa=1}^k \mathrm{rk}_\kappa$ by showing that a family of column vectors of V of cardinal $\sum_{\kappa=1}^k \mathrm{rk}_\kappa$ is linearly independent.

For all $\xi \in \mathbb{D}^e$, let $\mathcal{J}'(\xi) \subset \mathcal{J}(\xi)$ such that $\#\mathcal{J}'(\xi) = \mathrm{rk}(\xi)$ and the family $(V^j_\xi)_{j \in \mathcal{J}'(\xi)}$ is linearly independent. For all $\kappa = 1, \ldots, k$, $\mathcal{J}'_\kappa = \cup_{\xi \in \mathbb{D}^e_{\tau_\kappa}} \mathcal{J}'(\xi)$ and $\mathcal{J}' = \cup_{\kappa=1}^k \mathcal{J}'_\kappa$. We now prove that the family $(V^j)_{j \in \mathcal{J}'}$ is linearly independent.

Let $(\alpha_j) \in \mathbb{R}^{\mathcal{J}'}$ such that $\sum_{j\in\mathcal{J}'} \alpha_j V^j = 0$. We work by backward induction on κ from k to 1.

For all $\xi \in \mathbb{D}^e_{\tau_k}$, $\sum_{j\in\mathcal{J}'} \alpha_j V^j_{\mathbb{D}^+(\xi)} = 0$. Since $\tau_\kappa < \tau_k$ for all $\kappa = 1, \dots, k-1$, for all j such that $\xi(j) \notin \mathbb{D}^-(\xi) \cup \{\xi\}$, $V^j_{\mathbb{D}^+(\xi)} = 0$. So, one gets

$$\sum_{j\in\mathcal{J}'(\xi)} \alpha_j V^j_{\mathbb{D}^+(\xi)} + \sum_{\xi'\in\mathbb{D}^-(\xi)} \sum_{j\in\mathcal{J}'(\xi')} \alpha_j V^j_{\mathbb{D}^+(\xi)} = 0$$

From Assumption **(R)**,

$$\text{Vect}\left(V^{\mathcal{J}\left(\mathbb{D}^-(\xi)\right)}_{\mathbb{D}^+(\xi)}\right) \bigcap \text{Vect}\left(V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi)}\right) = \{0\}.$$

From the above equality,

$$\sum_{j\in\mathcal{J}'(\xi)} \alpha_j V^j_{\mathbb{D}^+(\xi)} \in \text{Vect}\left(V^{\mathcal{J}\left(\mathbb{D}^-(\xi)\right)}_{\mathbb{D}^+(\xi)}\right) \bigcap \text{Vect}\left(V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi)}\right)$$

hence $\sum_{j\in\mathcal{J}'(\xi)} \alpha_j V^j_{\mathbb{D}^+(\xi)} = 0$.

By construction, the family $(V^j_\xi)_{j\in\mathcal{J}')}$ is linearly independent and for all $\xi' \notin \mathbb{D}^+(\xi)$, $V^j_{\xi'} = 0$, so the family $(V^j_{\mathbb{D}^+(\xi)})_{j\in\mathcal{J}'(\xi)}$ is linearly independent. Hence, from above, one deduces that $\alpha_j = 0$ for all $j \in \mathcal{J}'(\xi)$. Since this is true for all $\xi \in \mathbb{D}^e_{\tau_k}$, one gets $\alpha_j = 0$ for all $j \in \mathcal{J}'_k$.

If $k = 2$, we are done since we have prove in Step 2 that the subspaces $\left(\text{Vect}\left(V^{\mathcal{J}(\xi)}\right)\right)_{\xi\in\mathbb{D}^e_{\tau_1}}$ are in direct sum, so the family $\left(V^j\right)_{j\in\mathcal{J}'_1}$ is linearly independent, hence for all $j \in \mathcal{J}'_1$, $\alpha_j = 0$.

If $k > 2$, we do again the same argument as above. Indeed, since we have proved that for all $j \in \mathcal{J}'_k$, $\alpha_j = 0$, for all $\xi \in \mathbb{D}^e_{\tau_{k-1}}$, $\sum_{j\in\mathcal{J}'} \alpha_j V^j_{\mathbb{D}^+(\xi)} = 0$ implies

$$\sum_{j\in\mathcal{J}'(\xi)} \alpha_j V^j_{\mathbb{D}^+(\xi)} + \sum_{\xi'\in\mathbb{D}^-(\xi)} \sum_{j\in\mathcal{J}'(\xi')} \alpha_j V^j_{\mathbb{D}^+(\xi)} = 0.$$

Using again Assumption **(R)**, one then deduces that for all $j \in \mathcal{J}'_{k-1}$, $\alpha_j = 0$.

Consequently, after a finite number of steps, we show that all α_j are equal to 0, which implies that the family $(V^j)_{j\in\mathcal{J}'}$ is linearly independent.

For the second part concerning the matrix $W(q)$ the proof is identical since for all $\xi \in \mathbb{D}^e$, the family $(W^j(q))_{j\in\mathcal{J}'(\xi)}$ is linearly independent and for all $j \in \mathcal{J}'(\xi) \cup \left(\cup_{\xi'\in\mathbb{D}^-(\xi)} J'(\xi')\right)$, $V^j_{\mathbb{D}^+(\xi)} = W^j_{\mathbb{D}^+(\xi)}(q)$. $\square$

Remark 4. *If the price q exhibits an arbitrage, then even under Assumption* **(R)***, the rank of V and the rank of $W(q)$ may be different. With a three dates economy where $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}\}$, two assets issued at ξ_0 and one* *asset issued at ξ_1, the asset price $q = \left(1, \frac{3}{2}, 1\right)$, then*

$$V = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad \mathbf{W}(\mathbf{q}) = \begin{bmatrix} -1 & -\frac{3}{2} & 0 \\ 1 & 2 & -1 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We note that $\mathrm{rank}V = 2 < \mathrm{rank}W(q) = 3$. *Nevertheless, the following result shows that if the payoff vectors are not redundant at each node, then the equality of ranks holds true even with an arbitrage price.*

Proposition 3. *Let us assume that V satisfies Assumption* **(R)**.

1) For all price $q \in \mathbb{R}^{\mathcal{J}}$, $\mathrm{rank}V \leq \mathrm{rank}W(q)$.

2) Furthermore, if for all $\xi \in \mathbb{D}^e$, $\mathrm{rank}V^{\mathcal{J}(\xi)} = n(\xi)$*, the number of assets issued at this node, then* $\mathrm{rank}V = \mathrm{rank}W(q)$ *for all price $q \in \mathbb{R}^{\mathcal{J}}$.*

Proof. 1) The proof is just an adaptation of the proof of Proposition 2. In the first step, since the price q is not supposed to be a non-arbitrage price, we get $\mathrm{rank}W^{\mathcal{J}(\xi)}(q) \geq \mathrm{rk}(\xi)$ instead of an equality. For the two next steps, the proofs never uses the fact that q is a non-arbitrage price, so we can replicate them to obtain $\mathrm{rank}W(q) \geq \sum_{\xi \in \mathbb{D}^e} \mathrm{rk}(\xi) = \mathrm{rank}V$.

2) If $\mathrm{rk}(\xi) = n(\xi)$ for all ξ, then $\sum_{\xi \in \mathbb{D}^e} \mathrm{rk}(\xi)$ is the cardinal of $\mathcal{J}$, which is the number of column of the matrix $W(q)$. So $\mathrm{rank}W(q) \leq \sum_{\xi \in \mathbb{D}^e} \mathrm{rk}(\xi) = \mathrm{rank}V$. □

If the assets issued at each node are linearly independent, then we can also obtain the equality of the rank under a slightly weaker assumption than Assumption **(R)** where we only deal with the returns at the immediate successors of a node ξ instead of looking at the whole returns for all successors.

Corollary 1. *Let us assume that:*

1) $\forall \xi \in \mathbb{D}^e$, $\mathrm{rank}V_{\xi^+}^{\mathcal{J}(\xi)} = n(\xi)$

and

2) $\mathit{Vect}\left(V_{\xi^+}^{\mathcal{J}\left(\mathbb{D}^-(\xi)\right)}\right) \bigcap \mathit{Vect}\left(V_{\xi^+}^{\mathcal{J}(\xi)}\right) = \{0\}$.

Then

$\mathrm{rank}V = \mathrm{rank}W(q)$.

Proof. We show that the assumptions of Proposition 3 are satisfied. First, we remark that $V_{\xi^+}^{\mathcal{J}(\xi)}$ is a sub-matrix of $V^{\mathcal{J}(\xi)}$, so $n(\xi) = \mathrm{rank}V_{\xi^+}^{\mathcal{J}(\xi)} \leq \mathrm{rank}V^{\mathcal{J}(\xi)}$. On the other hand, $\mathrm{rank}V^{\mathcal{J}(\xi)} \leq n(\xi)$ since the number of column of $V^{\mathcal{J}(\xi)}$ is $n(\xi)$. Hence $n(\xi) = \mathrm{rank}V^{\mathcal{J}(\xi)}$.

We now prove that Assumption **(R)** is satisfied. Let $\kappa \in \{2, \ldots, k\}$, $\xi \in \mathbb{D}^e_{\tau_\kappa}$ and $y \in \mathbb{R}^{\mathbb{D}^+(\xi)} \setminus \{0\}$ such that

$$y \in \mathrm{Vect}\left(V_{\mathbb{D}^+(\xi)}^{\mathcal{J}\left(\mathbb{D}^-(\xi)\right)}\right) \cap \mathrm{Vect}\left(V_{\mathbb{D}^+(\xi)}^{\mathcal{J}(\xi)}\right)$$

Then, there exists $(a_j) \in \mathbb{R}^{\mathcal{J}(\xi)}$ such that $y = \sum_{j\in\mathcal{J}(\xi)} a_j V^j_{\mathbb{D}^+(\xi)}$ and there exists $(b_j) \in \mathbb{R}^{\mathcal{J}(\mathbb{D}^-(\xi))}$ such that $y = \sum_{j\in\mathcal{J}(\mathbb{D}^-(\xi))} b_j V^j_{\mathbb{D}^+(\xi)}$. Restricting the above equality to the coordinates in ξ^+, one gets $y_{\xi^+} = \sum_{j\in\mathcal{J}(\xi)} a_j V^j_{\xi^+} = \sum_{j\in\mathcal{J}(\mathbb{D}^-(\xi))} b_j V^j_{\xi^+}$. From our second assumption, this implies that $y_{\xi^+} = 0$. From the first assumption, since the vectors $(V^j_{\xi^+})_{j\in\mathcal{J}(\xi)}$ are of maximal rank hence linearly independent, this implies that $a_j = 0$ for all $j \in \mathcal{J}(\xi)$. Hence, $y = 0$, which proves that $\mathrm{Vect}\left(V^{\mathcal{J}(\mathbb{D}^-(\xi))}_{\mathbb{D}^+(\xi)}\right) \bigcap \mathrm{Vect}\left(V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi)}\right) = \{0\}$. Consequently Assumption **(R)** is satisfied. □

3.2 Equality between the kernel of matrix of payoffs

In this subsection we prove a stronger result on the relationship between the payoff matrices. Proposition 4 states the equality of kerrnels for the payoff matrices under Assumption **(R)**. Although Proposition 2 is a consequence of Proposition 4, the proof of Proposition 4 is based upon the proof of Proposition 2.

Proposition 4. *If the payoff matrix V satisfies Assumption* **(R)***, then for all arbitrage free price $q \in \mathbb{R}^{\mathcal{J}}$,* $\mathrm{Ker}V = \mathrm{Ker}W(q)$[3].

Proof of Proposition 4. Let q be an arbitrage free price and let $\lambda = (\lambda_\xi) \in \mathbb{R}^{\mathbb{D}}_{++}$ such that ${}^tW(q)\lambda = 0$. From Proposition 2, $\mathrm{rank}V = \mathrm{rank}W(q)$ and this implies that $\dim\mathrm{Ker}V = \dim\mathrm{Ker}W(q)$. So, to get the equality of the kernels, it remains to show $\mathrm{Ker}V \subset \mathrm{Ker}W(q)$.

Let $z = (z_j)_{j\in\mathcal{J}} \in \mathbb{R}^{\mathcal{J}}$ be an element of the kernel of the payoff matrice V. So, $\sum_{j\in\mathcal{J}} z_j V^j = 0$ which is equivalent to: for all $\xi \in \mathbb{D}, \sum_{j\in\mathcal{J}} z_j V^j_\xi = 0$. Let us show that $z \in \mathrm{Ker}W(q)$. We work by backward induction on $\kappa \in \{1, ..., k\}$.

For all $\xi \in \mathbb{D}^e_{\tau_k}$, $\sum_{j\in\mathcal{J}} z_j V^j = 0$ implies that $\sum_{j\in\mathcal{J}} z_j V^j_{\mathbb{D}^+(\xi)} = 0$. For all j such that $\xi(j) \notin \mathbb{D}^-(\xi)$, $V^j_{\mathbb{D}^+(\xi)} = 0$. So we deduce that

$$\sum_{j\in\mathcal{J}(\xi)} z_j V^j_{\mathbb{D}^+(\xi)} + \sum_{\xi'\in\mathbb{D}^-(\xi)} \sum_{j\in\mathcal{J}(\xi')} z_j V^j_{\mathbb{D}^+(\xi)} = 0$$

From Assumption **(R)**, $\mathrm{Vect}\left(V^{\mathcal{J}(\mathbb{D}^-(\xi))}_{\mathbb{D}^+(\xi)}\right) \bigcap \mathrm{Vect}\left(V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi)}\right) = \{0\}$. From the above equality, $\sum_{j\in\mathcal{J}(\xi)} z_j V^j_{\mathbb{D}^+(\xi)}$ belongs to $\mathrm{Vect}\left(V^{\mathcal{J}(\mathbb{D}^-(\xi))}_{\mathbb{D}^+(\xi)}\right) \bigcap \mathrm{Vect}\left(V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi)}\right)$, hence $\sum_{j\in\mathcal{J}(\xi)} z_j V^j_{\mathbb{D}^+(\xi)} = 0$.

Moreover, by the fact that $V^j_{\xi'} = 0$ for all $\xi' \notin \mathbb{D}^+(\xi)$, one deduces that $\sum_{j\in\mathcal{J}(\xi)} z_j V^j = 0$.

For all $j \in \mathcal{J}(\xi)$ and for all $\eta \in \mathbb{D} \setminus \{\xi\}$, $V^j_\eta = W^j_\eta(q)$. At the node ξ, $\sum_{j\in\mathcal{J}(\xi)} z_j W^j_\xi(q) = -\sum_{j\in\mathcal{J}(\xi)} z_j q_j$. But $q_j = (1/\lambda_\xi) \sum_{\xi'\in\mathbb{D}^+(\xi)} \lambda_{\xi'} V^j_{\xi'}$. Hence,

[3] Let A be a linear map from $\mathbb{R}^n$ to $\mathbb{R}^p$. We denote its kernel by $\mathrm{Ker}A := \{z \in \mathbb{R}^n | Az = 0\}$.

$$\begin{aligned}\sum_{j\in\mathcal{J}(\xi)} z_j q_j &= (1/\lambda_\xi)\sum_{j\in\mathcal{J}(\xi)} z_j\left[\sum_{\xi'\in\mathbb{D}^+(\xi)}\lambda_{\xi'}V^j_{\xi'}\right]\\ &= (1/\lambda_\xi)\sum_{\xi'\in\mathbb{D}^+(\xi)}\lambda_{\xi'}\left[\sum_{j\in\mathcal{J}(\xi)} z_j V^j_{\xi'}\right]\\ &= 0\end{aligned}$$

So, we have proved that $\sum_{j\in\mathcal{J}(\xi)} z_j W^j(q) = 0$, and since it holds true for all $\xi \in \mathbb{D}^e_{\tau_k}$, $\sum_{\xi\in\mathbb{D}^e_{\tau_k}}\sum_{j\in\mathcal{J}(\xi)} z_j W^j(q) = 0$.

It remains to get that $\sum_{\kappa\in\{1,\dots,k-1\}}\sum_{\xi\in\mathbb{D}^e_{\tau_\kappa}}\sum_{j\in\mathcal{J}(\xi)} z_j W^j(q) = 0$. But we can do again the same argument for the nodes $\xi \in \mathbb{D}^e_{\tau_{k-1}}$ since for all $j \in \cup_{\kappa=1}^{k-1}\mathbb{D}^e_{\tau_\kappa}$, $V^j_{\mathbb{D}^+(\xi)} = 0$ if $\xi(j) \notin \mathbb{D}^-(\xi)$. Hence, we proved that $\sum_{j\in\mathcal{J}} z_j W^j(q) = 0$ that is $z \in \mathrm{Ker}W(q)$. $\square$

Remark 5. *If Assumption* **(R)** *is not satisfied, we may not have the conclusion of the Proposition 4. Let us consider a stochastic economy with $T = 2$ and three nodes, namely $\mathbb{D} = \{0, 1, 2\}$, and two assets j_1, j_2, where j_1 is issued at node 0 and pays -1 a node 1, 1 at node 2, j_2 is issued at node 1 and pays 1 at node 2. Consider the asset price $q = (0, 1)$; then the payoff matrices are*

$$V = \begin{bmatrix} 0 & 0\\ -1 & 0\\ 1 & 1\end{bmatrix} \quad , \quad W(q) = \begin{bmatrix} 0 & 0\\ -1 & -1\\ 1 & 1\end{bmatrix}$$

Take $z = (1, -1)$ we have $z \in \mathrm{Ker}W(q)$ and $z \notin \mathrm{Ker}V$

The following corollary is a generalization of Proposition 2.

Corollary 2. *Let $\mathcal{F} = \left(\mathcal{J}, (Z_i)_{i\in\mathcal{I}}\,(\xi(j))_{j\in\mathcal{J}}, V\right)$ be a financial structure such that Assumption* **(R)** *is satisfied and let G be a linear subspace of $\mathbb{R}^{\mathcal{J}}$. Then for all arbitrage free price q, $\dim[W(q)G] = \dim(VG)$.*

If $G = \mathbb{R}^{\mathcal{J}}$, the conclusion is the same as the one of Proposition 2. The proof of Corollary 2 is deduced from Proposition 4 and the following lemma, the proof of which is given in Appendix.

Lemma 2. *Let E and F be two vector spaces and φ and ψ be two linear maps from E to F then $\mathrm{Ker}\varphi = \mathrm{Ker}\psi$ if and only if for all linear subspace G of E, $\dim\varphi(G) = \dim\psi(G)$.*

4 Existence of equilibrium

4.1 The stochastic exchange economy

At each node $\xi \in \mathbb{D}$, there is a spot market on which a finite set $\mathbb{H} = \{1, \dots, H\}$ of divisible and physical goods are exchanged. We assume that each good is perishable, that is, its life does not have more than one date. In this model, a commodity is a pair $(h\xi)$ of a physical good $h \in \mathbb{H}$ and the node $\xi \in \mathbb{D}$ at which the good is available. Then the commodity space is $\mathbb{R}^{\mathbb{L}}$, where $\mathbb{L} = \mathbb{H} \times \mathbb{D}$. An

element $x \in \mathbb{R}^{\mathbb{L}}$ is called a consumption, that is to say $x = (x(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^{\mathbb{L}}$, where $x(\xi) = (x(h\xi))_{h \in \mathbb{H}} \in \mathbb{R}^{\mathbb{H}}$ for each $\xi \in \mathbb{D}$.
We denote by $p = (p(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^{\mathbb{L}}$ the vector of spot prices and $p(\xi) = (p(h\xi))_{h \in \mathbb{H}} \in \mathbb{R}^{\mathbb{H}}$ is called the spot price at node ξ. The spot price $p(h\xi)$ is the price at the node ξ for immediate delivery of one unit of the physical good h. Thus the value of a consumption $x(\xi)$ at node $\xi \in \mathbb{D}$ (measured in unit account of the node ξ) is

$$p(\xi) \bullet_{\mathbb{H}} x(\xi) = \sum_{h \in \mathbb{H}} p(h\xi)\, x(h\xi).$$

Each agent $i \in \mathcal{I}$ has a consumption set $X_i \subset \mathbb{R}^{\mathbb{L}}$, which consists of all possible consumptions. An allocation is an element $x \in \prod_{i \in \mathcal{I}} X_i$ and we denote by x_i the consumption of agent i, which is the projection of x on X_i.

The tastes of each consumer $i \in \mathcal{I}$ are represented by a *strict preference correspondence* $P_i : \prod_{j \in \mathcal{I}} X^j \longrightarrow X_i$, where $P_i(x)$ defines the set of consumptions that are strictly preferred to x_i for agent i, given the consumption x_j for the other consumers $j \neq i$. P_i represents the consumer tastes, but also his behavior with respect to time and uncertainty, especially his impatience and attitude toward risk. If consumer preferences are represented by utility functions $u_i : X_i \longrightarrow \mathbb{R}$ for each $i \in \mathcal{I}$, the strict preference correspondence is defined by $P_i(x) = \{\bar{x}_i \in X_i | u_i(\bar{x}_i) > u_i(x_i)\}$.

Finally, for each node $\xi \in \mathbb{D}$, every consumer $i \in \mathcal{I}$ has a node endowment $e_i(\xi) \in \mathbb{R}^{\mathbb{H}}$ (contingent on the fact that ξ prevails) and we denote by $e_i = (e_i(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^{\mathbb{L}}$ the endowments for the whole set of nodes. The exchange economy Σ can be summarized by

$$\Sigma = \left[\mathbb{D}, \mathbb{H}, \mathcal{I}, (X_i, P_i, e_i)_{i \in \mathcal{I}}\right].$$

4.2 Financial equilibrium

We now consider a financial exchange economy, which is defined as the couple of an exchange economy Σ and a financial structure $\mathcal{F}$. It can thus be summarized by

$$(\Sigma, \mathcal{F}) := \left[\mathbb{D}, \mathbb{H}, \mathcal{I}, \left(X_i, P^i, e_i\right)_{i \in \mathcal{I}}, \mathcal{J}, (Z_i)_{i \in \mathcal{I}}, (\xi(j))_{j \in \mathcal{J}}, V\right].$$

Given the price $(p, q) \in \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$, the budget set of consumer $i \in \mathcal{I}$ is $B^i_{\mathcal{F}}(p, q)$ defined by[4]:

$$\{(x_i, z_i) \in X_i \times Z_i : \forall \xi \in \mathbb{D}, p(\xi) \bullet_{\mathbb{H}} [x_i(\xi) - e_i(\xi)] \leq [W(p, q)\, z_i](\xi)\}$$

or

$$\{(x_i, z_i) \in X_i \times Z_i : \; p \diamond (x_i - e_i) \leq W(p, q)\, z_i\}.$$

We now introduce the equilibrium notion:

[4]For $x = (x(\xi))_{\xi \in \mathbb{D}}, p = (p(\xi))_{\xi \in \mathbb{D}}$ in $\mathbb{R}^{\mathbb{L}} = \mathbb{R}^{\mathbb{H} \times \mathbb{D}}$ (with $x(\xi)$, $p(\xi)$ in $\mathbb{R}^{\mathbb{H}}$) we let $p \diamond x = (p(\xi) \bullet_{\mathbb{H}} x(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^{\mathbb{D}}$.

Definition 1. *An equilibrium (resp. account clearing equilibrium) of the financial exchange economy* $(\Sigma, \mathcal{F})$ *is a list of strategies and prices* $(\bar{x}, \bar{z}, \bar{p}, \bar{q}) \in \left(\mathbb{R}^{\mathbb{L}}\right)^{\mathcal{I}} \times \prod_{i \in \mathcal{I}} Z_i \times \mathbb{R}^{\mathbb{L}} \setminus \{0\} \times \mathbb{R}^{\mathcal{J}}$ *such that*

(a) *for every* $i \in \mathcal{I}, (\bar{x}_i, \bar{z}_i)$ *maximizes the preferences* P^i *in the budget set* $B^i_{\mathcal{F}}(\bar{p}, \bar{q})$*, in the sense that*

$$(\bar{x}_i, \bar{z}_i) \in B^i_{\mathcal{F}}(\bar{p}, \bar{q}) \text{ and } \left[P^i(\bar{x}) \times Z_i\right] \bigcap B^i_{\mathcal{F}}(\bar{p}, \bar{q}) = \emptyset;$$

(b) $\sum_{i \in \mathcal{I}} \bar{x}_i = \sum_{i \in \mathcal{I}} e^i$ *(Commodity market clearing condition);*

(c) $\sum_{i \in \mathcal{I}} \bar{z}_i = 0$ *(resp.* $\sum_{i \in \mathcal{I}} W(\bar{q})\bar{z}_i = 0$ *) [Portfolio (resp. account) clearing condition)].*

An accounts clearing equilibrium only requires that the payoffs (or accounts) of the financial markets are cleared, that is $\sum_{i \in \mathcal{I}} W(\bar{q})\bar{z}_i = 0$, which is weaker than the portfolio clearing condition: $\sum_{i \in \mathcal{I}} \bar{z}_i = 0$. Every equilibrium of the economy $(\Sigma, \mathcal{F})$ is an accounts clearing equilibrium of $(\Sigma, \mathcal{F})$. The converse relationship between the two equilibrium notions is given in the next Corollary deduced from Proposition 1 of Cornet-Gopalan [3].

Corollary 3. *Let* $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ *be an account clearing equilibrium of* $(\Sigma, \mathcal{F})$ *satisfying one of the following conditions: (i)* $\cup_{i \in \mathcal{I}} Z_i$ *is a vector space, or (ii) Assumption* **(R)** *is satisfied and* $\mathrm{Ker} V \subset \cup_{i \in \mathcal{I}} Z_i$*, then there exists an equilibrium* $(\bar{x}, \hat{z}, \bar{p}, \bar{q})$ *of* $(\Sigma, \mathcal{F})$ *which difffers only in terms of the portfolio profile.*

4.3 No-arbitrage and financial equilibrium

Angeloni-Cornet[1] noted that when portfolios may be constrained, the concept of no-arbitrage has to be suitably modified. In particular, we shall make a distinction between the definitions of arbitrage-free portfolio and arbitrage-free financial structure.

Definition 2. *Given the financial structure* $\mathcal{F} = \left(\mathcal{J}, (Z_i)_{i \in \mathcal{I}}, (\xi(j))_{j \in \mathcal{J}}, V\right)$*, the portfolio* $\bar{z}_i \in Z_i$ *is said with no arbitrage opportunities or to be arbitrage-free for agent* $i \in \mathcal{I}$ *at the price* $(p, q) \in \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$ *if there is no portfolio* $z_i \in Z_i$ *such that* $W(p,q) z_i > W(p,q)(\bar{z}_i)$*, that is,* $[W(p,q) z_i](\xi) \geq [W(p,q)(\bar{z}_i)](\xi)$*, for every* $\xi \in \mathbb{D}$*, with at least one strict inequality,or, equivalently, if:*

$$W(p,q)(Z_i - \bar{z}_i) \cap \mathbb{R}^{\mathbb{D}}_+ = \{0\}.$$

The financial structure is said to be arbitrage-free at (p,q) *if there exists no portfolio* $(z_i) \in \prod_{i \in \mathcal{I}} Z_i$ *such that* $W(p,q)\left(\sum_{i \in \mathcal{I}} z_i\right) > 0$*, or, equivalently, if:*

$$W(p,q)\left(\sum_{i \in \mathcal{I}} Z_i\right) \cap \mathbb{R}^{\mathbb{D}}_+ = \{0\}.$$

Let the financial structure $\mathcal{F}$ be arbitrage-free at (p,q), and let $(\bar{z}_i) \in \prod_{i \in \mathcal{I}} Z_i$ such that $\sum_{i \in \mathcal{I}} \bar{z}_i = 0$. Then, for every $i \in \mathcal{I}$, $\bar{z}_i$ is arbitrage-free at (p, q). The converse is true, for example, when some agent's portfolio set is unconstrained, that is, when $Z_i = \mathbb{R}^{\mathcal{J}}$ for some $i \in \mathcal{I}$.

We recall that equilibrium portfolios are arbitrage-free under the following non-satiation assumption:

Assumption NS

(i) (Non-Saturation at Every Node.) For every $\bar{x} \in \prod_{i\in\mathcal{I}} X_i$ if $\sum_{i\in\mathcal{I}} \bar{x}_i = \sum_{i\in\mathcal{I}} e_i$, then, for every $i \in \mathcal{I}$, for every $\xi \in \mathbb{D}$, there exists $x_i \in X_i$ such that, for each $\xi' \neq \xi$, $x_i(\xi') = \bar{x}_i(\xi')$ and $x_i \in P^i(\bar{x})$;

(ii) if $x_i \in P^i(\bar{x})$, then $]\bar{x}_i, x_i] \subset P^i(\bar{x})$.

Proposition 5. *Under Assumption (NS), if $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium of the economy $(\Sigma, \mathcal{F})$, then $\bar{z}_i$ is no-arbitrage at price $(\bar{p}, \bar{q})$ for every $i \in \mathcal{I}$.*

The proof is given in Angeloni-Cornet [1] as well as the following characterization.

Theorem 1. *Let $\mathcal{F} = \left(\mathcal{J}, (Z_i)_{i\in\mathcal{I}}, (\xi(j))_{j\in\mathcal{J}}, V\right)$, let $(p, q) \in \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$, for $i \in \mathcal{I}$, let $z_i \in Z_i$, assume that Z_i is convex and consider the following statements:*

(i) *There exists $\lambda^i = \left(\lambda^i \xi)\right)_{\xi\in\mathbb{D}} \in \mathbb{R}^{\mathbb{D}}_{++}$ such that ${}^tW(p, q)\lambda^i \in N_{Z_i}(z_i)$[5], or, equivalently, there exists $\eta \in N_{Z_i}(z_i)$ such that for every $j \in \mathcal{J}$,*

$$\lambda^i_{\xi(j)} q_j = \sum_{\xi > \xi(j)} \lambda^i_\xi V^j_\xi(p) - \eta_j.$$

(ii) *The portfolio z_i is arbitrage free for agent $i \in \mathcal{I}$ at price (p, q).*

The implication $[(i) \Rightarrow (ii)]$ always holds and the converse is true under the additional assumption that Z_i is a polyhedral[6] set.

4.4 Existence of equilibrium

We introduce the following assumptions on the consumers and the financial structure. They are borrowed from Angeloni-Cornet [1] and Cornet-Gopalan [3]. In the following $\mathcal{Z}_{\mathcal{F}}$ is the linear space spanned by $\cup_{i\in\mathcal{I}} Z_i$. Since we only consider nominal assets, V does not depend on the spot price p.

Assumption (C) (Consumption Side) For all $i \in \mathcal{I}$ and all $\bar{x} \in \prod_{i\in\mathcal{I}} X_i$,

(i) X_i is a closed, convex and bounded below subset of $\mathbb{R}^{\mathbb{L}}$;

[5] we recall that $N_{Z_i}(z_i)$ is the normal cone to Z_i at z_i, which is defined as $N_{Z_i}(z_i) = \{\eta \in \mathbb{R}^{\mathcal{J}} : \eta \bullet_{\mathcal{J}} z_i \geq \eta \bullet_{\mathcal{J}} z'_i, \forall z'_i \in Z_i\}$.

[6] A subset $C \subset \mathbb{R}^n$ is said to be polyhedral if it is the intersection of finitely many closed half-spaces, namely $C = \{x \in \mathbb{R}^n : Ax \leq b\}$, where A is a real $(m \times n)$-matrix , and $b \in \mathbb{R}^m$. Note that polyhedral sets are always closed and convex and that the empty set and the whole space $\mathbb{R}^n$ are both polyhedral.

(ii) the preference correspondence P^i, from $\prod_{i\in\mathcal{I}} X_i$ to X_i, is lower semicontinuous[7] and $P^i(\bar{x})$ is convex;

(iii) for every $x_i \in P^i(\bar{x})$ for every $x_i' \in X_i$, $x_i' \neq x_i$, $[x_i', x_i[\cap P^i(\bar{x}) \neq \emptyset$[8];

(iv) (Irreflexivity) $\bar{x} \notin P^i(\bar{x})$;

(v) (Non-Saturation of Preferences at Every Node) if $\sum_{i\in\mathcal{I}} \bar{x}_i = \sum_{i\in\mathcal{I}} e_i$, for every $i \in \mathcal{I}$, for every $\xi \in \mathbb{D}$, there exists $x_i \in X_i$ such that, for each $\xi' \neq \xi$, $x_i(\xi') = \bar{x}_i(\xi')$ and $x_i \in P^i(\bar{x})$;

(vi) (Strong Survival Assumption) $e_i \in \mathrm{int}X_i$.

Note that these assumptions on P^i are satisfied when agents' preferences are represented by a continuous, strongly monotone and quasi-concave utility function.

Assumption (F) (Financial Side)

(i) for every $i \in \mathcal{I}$, Z_i is a closed, convex subset of $\mathbb{R}^{\mathcal{J}}$ containing 0;

(ii) there exists $i_0 \in \mathcal{I}$ such that $0 \in \mathrm{ri}_{\mathcal{Z}_{\mathcal{F}}}(Z_{i_0})$[9].

Note that we slightly weaken the assumption of Angeloni-Cornet [1] since we consider the linear space $\mathcal{Z}_{\mathcal{F}}$ instead of $\mathbb{R}^{\mathcal{J}}$ for the relative interior. Nevertheless, Assumption **(F)** is stronger than the corresponding one in Cornet-Gopalan [3] (Assumption **(FA)**), which is that the closed cone spanned by $\cup_{i\in\mathcal{I}} W(q)(Z_i)$ is a linear space. Indeed, if $0 \in \mathrm{ri}_{\mathcal{Z}_{\mathcal{F}}}(Z_{i_0})$, then the cone spanned by $W(q)(Z_{i_0})$ is equal to $W(q)(\mathcal{Z}_{\mathcal{F}})$, which is a linear space and since $Z_i \subset \mathcal{Z}_{\mathcal{F}}$ for all i, $W(q)(Z_i) \subset W(q)(\mathcal{Z}_{\mathcal{F}})$. Hence, $\cup_{i\in\mathcal{I}} W(q)(Z_i) = W(q)(\mathcal{Z}_{\mathcal{F}})$, which is a linear space.

In the following, we will consider the linear subspace G of $\mathcal{Z}_{\mathcal{F}}$ defined by:

$$G = \{u \in \mathcal{Z}_{\mathcal{F}} \mid \forall v \in \mathcal{Z}_{\mathcal{F}} \cap \mathrm{Ker}V, u \bullet_{\mathcal{J}} v = 0\}$$

Note that $G \cap \mathrm{Ker}V = \{0\}$ and for all $u \in \mathcal{Z}_{\mathcal{F}}$, $\mathrm{proj}_G(u) - u \in \mathrm{Ker}V$, where proj_G is the orthogonal projection from $\mathcal{Z}$ on G. Our main existence result is the following:

Proposition 6. **(a)** *Let*

$$(\Sigma, \mathcal{F}) := \left[\mathbb{D}, \mathbb{H}, \mathcal{I}, \left(X_i, P^i, e_i\right)_{i\in\mathcal{I}}, \mathcal{J}, (Z_i)_{i\in\mathcal{I}}, (\xi(j))_{j\in\mathcal{J}}, V\right]$$

be a financial economy with nominal assets satisfying Assumptions ***(C)***, ***(R)***, **(F)** *and such that for all $i \in \mathcal{I}$, the orthogonal projection of Z_i on G is closed. Then, for any given $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$, there exists an accounts clearing equilibrium $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ of $(\Sigma, \mathcal{F})$.*

[7] A correspondence $\phi : X \longrightarrow Y$ is said lower semicontinuous at $x_0 \in X$ if, for every open set $V \subset Y$ such that $V \cap \phi(x_0)$ is nonempty, there exists a neighborhood U of x_0 in X such that, for all $x \in U$, $V \cap \phi(x)$ is nonempty. The correspondence ϕ is said to be lower semicotinuous if it is lower semicontinuous at each point of X.

[8] This is satisfied, in particular, when $P^i(\bar{x})$ is open in X_i (for its relative topology).

[9] Let Z a nonempty subset of $\mathbb{R}^{\mathcal{J}}$ and let H a subspace of $\mathbb{R}^{\mathcal{J}}$ such that $Z \subset H$. We call relative interior of Z with respect to H denoted $\mathrm{ri}_H(Z)$ the set $\{z \in \mathbb{R}^{\mathcal{J}} \mid \exists r > 0; B(z,r) \cap H \subset Z\}$.

(b) *If moreover,* $\cup_{i\in\mathcal{I}} Z_i = Z_{\mathcal{F}}$ *or* $\mathrm{Ker} V \subset \cup_{i\in\mathcal{I}} Z_i$ *there exists an equilibrium* $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ *of* $(\Sigma, \mathcal{F})$.

Remark 6. *If for all* i, Z_i *is polyhedral, then the orthogonal projection on* G *is closed, so the existence result holds true for polyhedral portfolio sets under Assumptions* **(C)**, **(R)** *and* **(F)**. *When* Z_i *is not polyhedral, the projection on* G *is closed when the intersection of the asymptotic cone of* Z_i *with the kernel of* V *is included in the lineality space of* Z_i *(see Theorem 9.1 in Rockafellar [5]).*

If $\mathcal{Z}_{\mathcal{F}} \cap \mathrm{Ker} V = \{0\}$, *then* $G = \mathcal{Z}_{\mathcal{F}}$, *so, the orthogonal projection of* Z_i *on* G *is* Z_i *itself and it is already assumed to be closed in Assumption* **(F)**. *In particular,* $\mathcal{Z}_{\mathcal{F}} \cap \mathrm{Ker} V = \{0\}$ *if* V *is one to one or equivalently if* $\mathrm{rank} V = \#\mathcal{J}$.

The proof of our existence result is based upon Theorem 3.1 of Angeloni-Cornet [1]. To state this theorem, we need to introduce the set $B(\lambda)$ of admissible consumptions and portfolios for a given state price $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$, that is, the set of consumption-portfolio pair $(x, z) \in \prod_{i\in\mathcal{I}} X_i \times \prod_{i\in\mathcal{I}} Z_i$ such that there exists a commodity-asset price pair $(p, q) \in \bar{B}_{\mathbb{L}}(0,1) \times \mathbb{R}^{\mathcal{J}}$ satisfying:

$$\begin{cases} {}^t W(p,q)\lambda \in \bar{B}_{\mathcal{J}}(0,1), \\ (x_i, z_i) \in B^i_{\mathcal{F}}(p,q)\, \forall i \in \mathcal{I}, \\ \sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} e_i, \\ \sum_{i\in\mathcal{I}} z_i = 0 \end{cases}$$

The existence result requires that the set $B(\lambda)$ is bounded. In [1], it is proved that this holds true if the assets are all short-term and $\mathrm{rank} V = \#\mathcal{J}$ or, if there are long-term assets, that $\mathrm{rank} W(p,q) = \#\mathcal{J}$ for all $(p, q, \eta) \in B_L(0,1) \times \mathbb{R}^{\mathcal{J}} \times B_J(0,1)$ such that ${}^t W(p,q)\lambda = \eta$. Note that $B(\lambda)$ may be not bounded under the assumptions of Proposition 6. Let us consider

$$(\Sigma, \mathcal{F}) := \left[\mathbb{D}, \mathbb{H}, \mathcal{I}, \left(X_i, P^i, e_i\right)_{i\in\mathcal{I}}, \mathcal{J}, (Z_i)_{i\in\mathcal{I}}, (\xi(j))_{j\in\mathcal{J}}, V\right]$$

a financial exchange economy with 3 nodes without uncertainty, satisfying assumptions **C** with $X_i = \mathbb{R}^3$, $\mathcal{I} = 2$, $\mathbb{H}$ is a singleton, $Z_i = \mathbb{R}^3$ and such that for every asset price $q = (q_1, q_2, q_3)$ the full payoff matrix is:

$$\mathbf{W}(\mathbf{q}) = \begin{bmatrix} -q_1 & 0 & 0 \\ 1 & -q_2 & -q_3 \\ 2 & 1 & 2 \end{bmatrix}$$

Let (z_i^n) be a sequence of elements of $\mathbb{R}^3$ such that

$$z_1^n = \begin{pmatrix} 1 \\ -2n \\ n \end{pmatrix} \quad \text{and} \quad z_2^n = \begin{pmatrix} -1 \\ 2n \\ -n \end{pmatrix}$$

Let us consider the arbitrage free price $q = \left(2, \frac{1}{2}, 1\right)$. The spot price is $p = (1,1,1)$. Let $e_1 = (3,3,3) = e_2$ and $\hat{x}_1 = (2,4,5)$ and $\hat{x}_2 = (4,2,1)$. Clearly, for all n, $(\hat{x}, z^n) \in B(\lambda)$ for $\lambda = \left(1, 1, \frac{1}{2}\right)$ since for all $n \in \mathbb{N}$, ${}^t[W(p,q)\, z_1^n] = (-1, 1, 2)$ and ${}^t[W(p,q)\, z_1^n] = (1, -1, -2)$. Hence $B(\lambda)$ is not bounded since (z^n) is not bounded.

Our contribution is to obtain an existence result with long-term assets with assumptions only on the fundamentals of the economy, namely the payoff matrix V and the portfolio sets Z_i, regardless of the arbitrage free price. Note that in Cornet-Gopalan [3], Assumption **(FA)** depends on the asset price q, which is an endogenous variable.

The proof of Proposition 6 is divided into three steps. We first prove that the set $B(\lambda)$ is bounded under an additional condition that for all i, $Z_i \subset G$, where G is a linear subspace of $\mathcal{Z}_{\mathcal{F}}$ satisfying $G \cap \mathrm{Ker} V = \{0\}$ (Proposition 7). Then, we deduce the existence of an equilibrium under this additional assumption from Theorem 3.1 of Angeloni-Cornet [1] with a slight adaptation of the proof to deal with the space $\mathcal{Z}_{\mathcal{F}}$ instead of $\mathbb{R}^{\mathcal{J}}$. Finally, we show how to deduce an equilibrium of the original economy from an equilibrium where the portfolio sets are the projection of the initial portfolio sets on the subspace G of $\mathcal{Z}_{\mathcal{F}}$ (Proposition 8).

Proposition 7. *Let*

$$(\Sigma, \mathcal{F}) := \left[\mathbb{D}, \mathbb{H}, \mathcal{I}, \left(X_i, P^i, e_i\right)_{i\in\mathcal{I}}, \mathcal{J}, (Z_i)_{i\in\mathcal{I}}, (\xi(j))_{j\in\mathcal{J}}, V\right]$$

s be a financial economy satisfying for all $i \in \mathcal{I}$, X_i is bounded below, $\mathcal{F}$ consists of nominal assets and satisfies Assumptions **(R)** *and* **(F)**. *If moreover for all $i \in \mathcal{I}$, Z_i is included in a linear subspace G such that $G \cap \mathrm{Ker}V = \{0\}$, then, for any given $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$, $B(\lambda)$ is bounded.*

The proof of Proposition 7 is in Appendix.

Proof of Proposition 6 *when for all $i \in \mathcal{I}$, Z_i is included in a linear subspace G such that $G \cap \mathrm{Ker}V = \{0\}$.*

Thanks to Proposition 7, all assumptions of Theorem 3.1 of Angeloni-Cornet [1] are satisfied but the fact that $0 \in \mathrm{ri}_{\mathcal{Z}_{\mathcal{F}}}(Z_{i_0})$ instead of $0 \in \mathrm{int} Z_{i_0}$. To complete the proof, we now show how to adapt the proof of Angeloni-Cornet to this slightly more general condition.

In the preliminary definitions, η is chosen in $\mathcal{Z}_{\mathcal{F}}$ instead of $\mathbb{R}^{\mathcal{J}}$. In Step 2 of the proof of Claim 4.1, if $\eta \neq 0$, we obtain $0 < \max\{\eta \bullet_{\mathcal{J}} z_{i_0} \mid z_{i_0} \in Z_{i_0}\}$ since Z_{i_0} is included in $\mathcal{Z}_{\mathcal{F}}$, $\eta \in \mathcal{Z}_{\mathcal{F}}$ and $0 \in \mathrm{ri}_{\mathcal{Z}_{\mathcal{F}}}(Z_{i_0})$, so $r\eta \in Z_{i_0}$ for $r > 0$ small enough. In Claim 4.3 of Sub-sub-section 4.1.3, the argument holds true since $\bar{z}_i \in \mathcal{Z}_{\mathcal{F}}$ for all i and so, $(1/\|\sum_{i\in\mathcal{I}} \bar{z}_i\|) \sum_{i\in\mathcal{I}} \bar{z}_i$ belongs to $\mathcal{Z}_{\mathcal{F}}$. In Sub-sub-section 4.2.2, to show that $0 \in \mathrm{ri}_{\mathcal{Z}_{\mathcal{F}}}(Z_{i_0 r})$ in the truncated economy, it suffices to remark that there exists $r' > 0$ such that $B_{\mathcal{J}}(0, r') \cap \mathcal{Z}_{\mathcal{F}} \subset Z_{i_0}$, hence, $B_{\mathcal{J}}(0, \min\{r, r'\}) \cap \mathcal{Z}_{\mathcal{F}} \subset Z_{i_0 r}$, which means that 0 belongs to the relative interior of $Z_{i_0 r}$ with respect to $\mathcal{Z}_{\mathcal{F}}$. □

To complete the proof of Proposition 6 in the general case, we consider the subspace G defined before the statement of Proposition 6 and we recall that $G \cap \mathrm{Ker}V = \{0\}$. We consider the financial economy

$$(\Sigma, \mathcal{F}') = \left[\mathbb{D}, \mathbb{H}, \mathcal{I}, \left(X_i, P^i, e_i\right)_{i\in\mathcal{I}}, \mathcal{J}, (Z'_i)_{i\in\mathcal{I}}, (\xi(j))_{j\in\mathcal{J}}, V\right]$$

with $Z'_i = \mathrm{proj}_G(Z_i)$ for every $i \in \mathcal{I}$. Since $0 \in \mathrm{ri}_{\mathcal{Z}_{\mathcal{F}}}(Z_{i_0})$, $0 \in \mathrm{ri}_G(Z'_{i_0})$. Furthermore, since $Z'_i \subset G$ for all i, one deduces that the linear space $\mathcal{Z}'$ spanned by the sets (Z'_i) is equal to G, that is $\mathcal{Z}_{\mathcal{F}'} = G$. Hence, $(\Sigma, \mathcal{F}')$ satisfies all assumptions of Proposition 6 and $Z'_i \subset G$ for all i and $G \cap \mathrm{Ker}V = \{0\}$. The end of the proof of Proposition 6 is a consequence of the following proposition, the proof of which is in Appendix.

Proposition 8. *Let $(\Sigma, \mathcal{F})$ be the financial economy of the Proposition 6. Let $(\Sigma, \mathcal{F}') = \left[\mathbb{D}, \mathbb{H}, \mathcal{I}, \left(X_i, P^i, e_i\right)_{i\in\mathcal{I}}, \mathcal{J}, (Z'_i)_{i\in\mathcal{I}}, (\xi(j))_{j\in\mathcal{J}}, V\right]$ be the financial economy such that $Z'_i = \mathrm{proj}_G(Z_i)$ for every $i \in \mathcal{I}$. If $(\bar{x}, \bar{z}', \bar{p}, \bar{q})$ is an equilibrium of financial economy $(\Sigma, \mathcal{F}')$ associated to the state price λ, then there exists $\bar{z}$ such that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an accounts clearing equilibrium of the financial economy $(\Sigma, \mathcal{F})$.*

5 Appendix

Proof of lemma 1. Let us denote by k [resp. k'] the number of dates where there are issuance of at least one asset for the financial structure $\mathcal{F}$ [resp. $\mathcal{F}'$]. It is clear that $k' \leq k$.

By Assumption **(R)**, we have: for all $\kappa \in \{2, \ldots, k\}$ and for all $\xi \in \mathbb{D}^e_{\tau_\kappa}$,

$$\mathrm{Vect}\left(V^{\mathcal{J}(\mathbb{D}^-(\xi))}_{\mathbb{D}^+(\xi)}\right) \bigcap \mathrm{Vect}\left(V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi)}\right) = \{0\}.$$

Since $\mathcal{J}' \subset \mathcal{J}$, $\mathrm{Vect}\left(V^{\mathcal{J}'((\mathbb{D}^-(\xi))}_{\mathbb{D}^+(\xi)}\right) \subset \mathrm{Vect}\left(V^{\mathcal{J}((\mathbb{D}^-(\xi))}_{\mathbb{D}^+(\xi)}\right)$ and $\mathrm{Vect}\left(V^{\mathcal{J}'(\xi)}_{\mathbb{D}^+(\xi)}\right) \subset \mathrm{Vect}\left(V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi)}\right)$. So,

$$\mathrm{Vect}\left(V^{\mathcal{J}'(\xi^-)}_{\mathbb{D}^+(\xi)}\right) \cap \mathrm{Vect}\left(V^{\mathcal{J}'(\xi)}_{\mathbb{D}^+(\xi)}\right) \subset \mathrm{Vect}\left(V^{\mathcal{J}(\mathbb{D}^-(\xi))}_{\mathbb{D}^+(\xi)}\right) \bigcap \mathrm{Vect}\left(V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi)}\right) = \{0\}.$$

hence the financial structure $\mathcal{F}'$ satisfies Assumption **(R)**. □

Proof of lemma 2. We first show that the equality of the kernels implies the equality of dimensions of the images. Let G be a linear subspace of E and let φ_G (resp. ψ_G) be the restriction of φ (resp. ψ) at G. We have $\varphi(G) = \mathrm{Im}\varphi_G$[10] and $\dim \mathrm{Im}\varphi_G = \dim G - \dim(\mathrm{Ker}\varphi_G)$. As $\mathrm{Ker}\varphi = \mathrm{Ker}\psi$ we have $\mathrm{Ker}\varphi_G = (\mathrm{Ker}\varphi) \cap G = (\mathrm{Ker}\psi) \cap G = \mathrm{Ker}\psi_G$ hence $\dim \varphi(G) = \dim \psi(G)$.

Let us show the converse implication. If $\mathrm{Ker}\varphi \neq \mathrm{Ker}\psi$, then there exists $u \in \mathrm{Ker}\varphi$ such that $u \notin \mathrm{Ker}\psi$ or there exists $u \in \mathrm{Ker}\psi$ such that $u \notin \mathrm{Ker}\varphi$. In the first case, with $G = \mathrm{Ker}\varphi$, we have $\varphi(G) = \{0\} \neq \psi(G)$, hence $\dim \varphi(G) = 0 < \dim \psi(G)$. In the second case, we obtain the same inequality with $G = \mathrm{Ker}\psi$. So the equality of the dimension of $\varphi(G)$ and $\psi(G)$ for all linear subspace G implies the equality of kernels. □

The following lemma will facilitate the proof of Proposition 7.

[10]Let γ be a linear map from E to F. We denote its image by $\mathrm{Im}\gamma := \{y \in F \mid \exists z \in E; y = \gamma(z)\}$.

Lemma 3. *Let A be a compact subset of $\mathbb{R}^N$. For all $\alpha \in A$, let $W(\alpha)$: $\mathbb{R}^{\mathcal{J}} \longrightarrow \mathbb{R}^{\mathbb{D}}$ be a linear mapping and G be a linear supspace of $\mathbb{R}^N$ such that* $\mathrm{Ker}W(\alpha) \cap G = \{0\}$ *for all $\alpha \in A$. If the application $\alpha \mapsto W(\alpha)$ is continuous, then there exists $c > 0$ such that:*

$$\|W(\alpha)z\| \geq c\,\|z\| \text{ for all } (z,\alpha) \in G \times A.$$

Proof of Lemma 3. By contradiction. Let us assume that, for every $n \in \mathbb{N}$, there exists $z_n \in G$ and $\alpha_n \in A$ such that $\|W(\alpha_n)\, z_n\| < \frac{1}{n}\|z_n\|$. Noticing that $z_n \neq 0$, without any loss of generality we can assume that $((1/\|z_n\|)z_n)_n$ (which is in the unit sphere of $\mathbb{R}^N$ and in the closed linear subspace G) converges to some element $v \in G \setminus \{0\}$ and (α_n) converges to some element $\alpha \in A$ (since A is compact). By continuity of the map W, taking the limit when $n \longrightarrow \infty$, we get $\|W(\alpha)v\| \leq 0$, hence $v \in \mathrm{Ker}W(\alpha)$, a contradiction with the fact that $\mathrm{Ker}W(\alpha) \cap G = \{0\}$. □

Proof of Proposition 7 For every $i \in \mathcal{I}, \lambda \in \mathbb{R}^{\mathbb{D}}_{++}$, we let $\hat{X}_i(\lambda)$ and $\hat{Z}_i(\lambda)$ be the projections of $B(\lambda)$ on X_i and Z_i, respectively, that is:

$$\hat{X}_i(\lambda) := \left\{ x_i \in X_i \mid \exists\, (x_j)_{j\neq i} \in \prod_{j\neq i} X_j, \exists z \in \prod_{i\in\mathcal{I}} Z_i, (x,z) \in B(\lambda) \right\}$$

$$\hat{Z}_i(\lambda) := \left\{ z_i \in Z_i \mid \exists\, (z_j)_{j\neq i} \in \prod_{j\neq i} Z_j,\ \exists x \in \prod_{i\in\mathcal{I}} X_i, (x,z) \in B(\lambda) \right\}.$$

It suffices to prove that $\hat{X}_i(\lambda)$ and $\hat{Z}_i(\lambda)$ are bounded sets for every i to show that $B(\lambda)$ is bounded

We first show that for every $i \in \mathcal{I}$ the set $\hat{X}_i(\lambda)$ is bounded. Indeed, since the sets X_i are bounded below, there exists $\underline{x}_i \in \mathbb{R}^{\mathbb{L}}$ such that $X_i \subset \{\underline{x}_i\} + \mathbb{R}^{\mathbb{L}}_+$. If $x_i \in \hat{X}_i(\lambda)$, there exists $x_j \in X_j$, for every $j \neq i$, such that $\sum_{j\in\mathcal{I}} x_j = \sum_{j\in\mathcal{I}} e_j$. Consequently,

$$\underline{x}'_i \leq x_i = -\sum_{j\neq i} x_j + \sum_{j\in\mathcal{I}} e_j \leq -\sum_{j\neq i} \underline{x}_j + \sum_{j\in\mathcal{I}} e_j$$

and so $\hat{X}_i(\lambda)$ is bounded.

We now show that $\hat{Z}_i(\lambda)$ is bounded. Indeed, for every $z_i \in \hat{Z}_i(\lambda)$, there exists $(z_j)_{j\neq i} \in \prod_{j\neq i} Z_j$, $x \in \prod_{j\in\mathcal{I}} X_j$, $p \in \bar{B}_{\mathbb{L}}(0,1)$, $q \in \mathbb{R}^{\mathcal{J}}$ such that ${}^tW(q)\lambda \in \bar{B}_{\mathcal{J}}(0,1)$, $\sum_{j\in\mathcal{I}} z_j = 0$ and $(x_j, z_j) \in B^j_{\mathcal{F}}(p,q)$ for every $j \in \mathcal{I}$. As $(x_i, z_i) \in B^i_{\mathcal{F}}(p,q)$ and $(x_i, p) \in \hat{X}_i(\lambda) \times \bar{B}_{\mathbb{L}}(0,1)$, a compact set, there exists $\alpha_i \in \mathbb{R}^{\mathbb{D}}$ such that

$$\alpha_i \leq p \diamond (x_i - e_i) \leq W(q)z_i.$$

Using the fact that $\sum_{i\in\mathcal{I}} z_i = 0$, we also have

$$W(q)z_i = W(q)\left(-\sum_{j\neq i} z_j\right) \leq -\sum_{j\neq i} \alpha_j,$$

hence there exists $r > 0$ such that $W(q)z_i \in \bar{B}_{\mathbb{D}}(0, r)$.

Since Assumption **(R)** holds true, $\mathrm{Ker}W(q) = \mathrm{Ker}V$ for all arbitrage free price $q \in \mathbb{R}^{\mathcal{J}}$. Hence $G \cap \mathrm{Ker}W(q) = \{0\}$ for all q. Lemma 3 applied to $W(q)$ for $q \in A := \left\{q \in \mathbb{R}^{\mathcal{J}} \mid {}^tW(q)\lambda \in \bar{B}_{\mathcal{J}}(0, 1)\right\}$, which is a compact subset of $\mathbb{R}^{\mathcal{J}}$, implies that there exists $c > 0$ such that, for every $q \in A$, $z_i \in G$, $c\,\|z_i\| \leq \|W(q)z_i\|$ hence

$$c\,\|z_i\| \leq \|W(q)z_i\| \leq r \text{ for every } z_i \in \hat{Z}_i(\lambda),$$

and this shows that the set $\hat{Z}_i(\lambda)$ is bounded. □

Proof of Proposition 8 Let $(\bar{x}, \bar{z}', \bar{p}, \bar{q})$ be an equilibrium of the financial exchange economy $(\Sigma, \mathcal{F}')$. For all $i \in \mathcal{I}$, let $\bar{z}_i \in Z_i$ such that $\bar{z}'_i = \mathrm{proj}_G(\bar{z}_i)$. We show that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an account clearing equilibrium of the financial exchange economy $(\Sigma, \mathcal{F})$.

Note that the definition of G as a subspace of $\mathcal{Z}_{\mathcal{F}}$ orthogonal to $\mathcal{Z} \cap \mathrm{Ker}V$, implies that $\mathrm{proj}_G(z_i) - z_i \in \mathrm{Ker}V$ for all i and for all $z_i \in \mathcal{Z}_{\mathcal{F}}$. Furthermore, Assumption **(R)** implies that $\mathrm{Ker}W(\bar{q}) = \mathrm{Ker}V$, so $\mathrm{proj}_G(z_i) - z_i \in \mathrm{Ker}W(\bar{q})$. Note also that the full payoff matrices of the financial structures $\mathcal{F}$ and $\mathcal{F}'$ are the same.

- $\forall i \in \mathcal{I}, (\bar{x}_i, \bar{z}_i) \in B^i_{\mathcal{F}}(\bar{p}, \bar{q})$. Indeed, from the definition of $\bar{z}_i$ and the above remark, we have:

  $$W(\bar{q})\bar{z}'_i = W(\bar{q})(\bar{z}'_i - \bar{z}_i) + W(\bar{q})\bar{z}_i = W(\bar{q})\bar{z}_i,$$

  so for all $\xi \in \mathbb{D}$,

  $$\bar{p}(\xi) \bullet_{\mathbb{H}} [\bar{x}_i(\xi) - e_i(\xi)] \leq [W(\bar{q})\bar{z}'_i](\xi) = [W(\bar{q})\bar{z}_i](\xi).$$

- $\forall i \in \mathcal{I}, \left[P^i(\bar{x}) \times Z_i\right] \bigcap B^i_{\mathcal{F}}(\bar{p}, \bar{q}) = \emptyset$. Let us argue by contradiction. If there exists $i \in \mathcal{I}$ and (x_i, z_i) such that $(x_i, z_i) \in \left[P^i(\bar{x}) \times Z_i\right] \bigcap B^i_{\mathcal{F}}(\bar{p}, \bar{q})$, this implies that $x_i \in P^i(\bar{x})$ and for all $\xi \in \mathbb{D}$, $\bar{p}(\xi) \bullet_{\mathbb{H}} [x_i(\xi) - e_i(\xi)] \leq [W(\bar{q})z_i](\xi)$. Let $z'_i = \mathrm{proj}_G(z_i)$. Note that $z'_i \in Z'_i$. Since $z'_i - z_i \in \mathrm{Ker}V = \mathrm{Ker}W(\bar{q})$,

  $$W(\bar{q})z_i = W(\bar{q})(z'_i + z_i - z'_i) = W(\bar{q})z'_i$$

  and this implies that

  $$\forall i \in \mathcal{I}, \forall \xi \in \mathbb{D}, \bar{p}(\xi) \bullet_{\mathbb{H}} [\bar{x}_i(\xi) - e_i(\xi)] \leq [W(\bar{q})z'_i](\xi)$$

  that is $(x_i, z'_i) \in B^i_{\mathcal{F}'}(\bar{p}, \bar{q})$. Since $(x_i, z'_i) \in \left[P^i(\bar{x}) \times Z'_i\right]$, this contradicts the fact that $(\bar{x}, \bar{z}', \bar{p}, \bar{q})$ is an equilibrium of the financial exchange economy $(\Sigma, \mathcal{F}')$.

- $\sum_{i \in \mathcal{I}} \bar{x}_i = \sum_{i \in \mathcal{I}} e^i$, from the market clearing condition at the equilibrium $(\bar{x}, \bar{z}', \bar{p}, \bar{q})$. $\sum_{i \in \mathcal{I}} \bar{z}'_i = 0$ from the market clearing condition on the asset markets. Since $\sum_{i \in \mathcal{I}} \bar{z}'_i = \mathrm{proj}_G(\sum_{i \in \mathcal{I}} \bar{z}_i)$,

  $$W(\bar{q})(\sum_{i \in \mathcal{I}} \bar{z}_i) = W(\bar{q})(\sum_{i \in \mathcal{I}} \bar{z}'_i) = 0.$$

  Hence, $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an account clearing equilibrium of the financial exchange economy $(\Sigma, \mathcal{F})$. □

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